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LETTER TO THE EDITOR

Renormalisation group for the transfer matrix

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Abstract. A real space renormalisation group method is developed for the transfer matrix, and applied to the anisotropic two-dimensional Ising model as an example. It is possible to maintain self-duality and so to obtain the exact critical line.

The equivalence between classical statistical mechanics systems in $(d+1)$ dimensions and zero-temperature quantum systems in d dimensions (Suzuki 1971) has been widely used to study properties of the former. Real space, block-spin, renormalisation group (RG) methods have been developed for the quantum Hamiltonian (Drell *et al* 1976, 1977, Pearson 1976, Cardy 1976, Jullien *et al* 1978, Hirsch and Mazenko 1979). In these methods, degrees of freedom are thinned only in the d transverse directions. Rescaling in the remaining 'time' direction corresponds to a rescaling of the energy scale of the Hamiltonian, which appears as a consequence of the calculation. Such methods should then be suitable for studying systems exhibiting modulated phases (Elliot 1961, Selke and Fisher 1979 and references therein, Ostlund 1981), for which an explicit rescaling in the direction of the modulation would be inappropriate.

However, the equivalence mentioned above is valid only in a highly anisotropic limit of the classical system (which corresponds to the time-continuum limit) in which any commensurate modulated phases will disappear. It is therefore desirable to develop RG methods directly for the transfer matrix without taking the anisotropic limit. In this letter we illustrate such a technique applied to the two-dimensional Ising model, although extensions to more dimensions or other systems are straightforward.

The transfer matrix for a nearest-neighbour model in two dimensions can be written

$$\hat{T} = \dots T_1 T_2 T_3 T_4 \dots V_{12} V_{23} V_{34} \dots \tag{1}$$

where T_j is the single site transfer matrix, and $V_{j,j+1}$ is the Boltzmann weight for a horizontal link. The basic idea is to form blocks whose transfer matrices commute, diagonalise these blocks exactly, retain only the highest-lying states in each block, and re-express the full transfer matrix in the basis of these states. In this truncated basis $\hat{T}$ has the same form as before, with renormalised parameters. In order to build blocks some of the matrices in (1) have to be brought together, and at this point their non-commutativity must be taken into account. Also, the block transfer matrices may not be Hermitian. These two difficulties do not appear in the Hamiltonian limit.

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As examples we exhibit two approximate RG schemes for the transfer matrix of the two-dimensional Ising model, for which

$$T_j = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \tag{2}$$

$$V_{j,j+1} = 1 + \lambda \sigma_j^x \sigma_{j+1}^x \tag{3}$$

where $\varepsilon = \tanh J_1$, $\lambda = \tanh J_0$, with J_1 and J_0 being the nearest-neighbour couplings in the 'time' and horizontal directions, respectively. First we consider blocks with $T_b = T_{2j} V_{2j,2j+1}$. This has the advantage of preserving duality. After a similarity transformation, the transfer matrix may be written

$$\hat{T}' = \ldots V_{12} V_{34} V_{56} \ldots T_1 T_3 T_5 \ldots (T_2 V_{23})(T_4 V_{45})(T_6 V_{67}) \ldots \tag{4}$$

where the blocks on the right will be treated exactly. T_b has two degenerate largest eigenvalues

$$\mu = \tfrac{1}{2}\{1 + \varepsilon + [(1-\varepsilon)^2 + 4\varepsilon\lambda^2]^{1/2}\} \tag{5}$$

whose right and left eigenstates have the form

$$|\uparrow\rangle_{\mathrm{R}} = a|\uparrow\uparrow\rangle + b|\downarrow\downarrow\rangle \qquad \langle\uparrow|_{\mathrm{L}} = a'\langle\uparrow\uparrow| + b'\langle\downarrow\downarrow| \tag{6}$$

and

$$|\downarrow\rangle_{\mathrm{R}} = a|\uparrow\downarrow\rangle + b|\downarrow\uparrow\rangle \qquad \langle\downarrow|_{\mathrm{L}} = a'\langle\uparrow\downarrow| + b'\langle\downarrow\uparrow| \tag{7}$$

where

$$a'/b' = \varepsilon a/b = \varepsilon\lambda/(\mu - 1). \tag{8}$$

The correct normalisation condition is $aa' + bb' = 1$. $\hat{T}'$ is now written in terms of the truncated basis formed by (6) and (7). In this basis the blocks on the right of (4) are proportional to the identity matrix, and give an unimportant additive constant in the free energy. The remaining factors T_{2j+1} and $V_{2j-1,2j}$ have the form of equations (2) and (3) respectively, with renormalised parameters

$$\varepsilon' = (\varepsilon a'a + b'b)/(a'a + \varepsilon b'b) \tag{9}$$

$$\lambda' = \lambda(ab' + a'b) \tag{10}$$

and then, after a further similarity transformation, the full transfer matrix may be recast in form (1). The flows of (9) and (10) in the (ε, λ) plane are shown in figure 1. In the paramagnetic phase, the flows terminate on a line of fixed points $(\varepsilon \neq 0, \lambda = 0)$. The final value of ε is (within our approximation) just the ratio of the second to the largest eigenvalue of $\hat{T}$, so the correlation length is $-(\ln \varepsilon)^{-1}$. In the ferromagnetic phase flows terminate on the line $(\varepsilon = 1, \lambda \neq 0)$. The two phases are separated by a critical line which, because our procedure preserves duality, is exact. The equation $\sinh 2J_1 \sinh 2J_0 = 1$ for the critical line becomes, in our variables,

$$\varepsilon\lambda + \varepsilon + \lambda = 1 \tag{11}$$

and this is preserved under the transformations (9) and (10). Points near the critical line flow into the region $\varepsilon \sim 1$, $\lambda \sim 0$. This is just the anisotropic limit in which the equivalence to a quantum model is valid. Defining $K = (1-\varepsilon)/\lambda$, the ratio of (9) and (10) becomes

$$K' = \tfrac{1}{2}K^2 \tag{12}$$

in this limit. This is the same equation as derived by Fernandez-Pacheco (1979) using a similar method in the Hamiltonian limit. Thus we obtain the same values for the critical exponents.

An alternative decomposition of $\hat{T}$ which we have considered is

$$\hat{T} = \ldots V_{23}^{1/2} V_{45}^{1/2} \ldots (V_{12}^{1/2} T_1 T_2 V_{12}^{1/2})(V_{34}^{1/2} T_3 T_4 V_{34}^{1/2}) \ldots V_{23}^{1/2} V_{45}^{1/2} \ldots \tag{13}$$

which is Hermitian. Now $T_b = V_{2j-1,2j}^{1/2} T_{2j-1} T_{2j} V_{2j-1,2j}^{1/2}$ is treated exactly. The flows are qualitatively similar to those of figure 1. The critical fixed point once again occurs in the anisotropic limit, and in this limit our equations reproduce those of Drell *et al* (1976).

Although the values of critical exponents obtained in these simple truncation schemes are not reliable, accuracy may be improved, as in the Hamiltonian case, by considering either larger blocks (Jullien *et al* 1978), or the effect of other eigenstates of the block transfer matrix (Hirsch and Mazenko 1979).

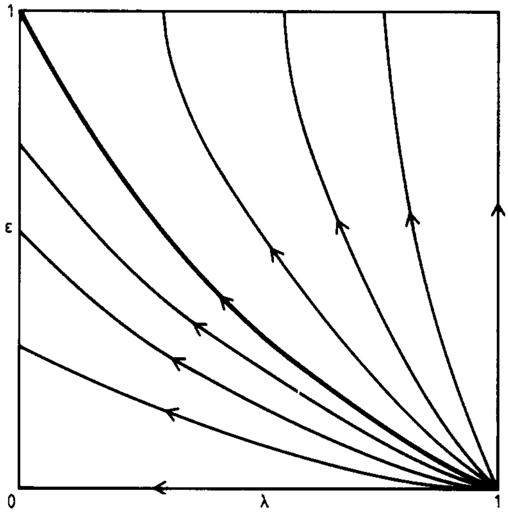

Figure 1. Schematic diagram of the RG flows for the transformations considered. The bold curve represents the critical line. Points on this line flow into the critical fixed point $\varepsilon = 1$, $\lambda = 0$.

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